Stresses Under Spread Footing Corners Due to Plane Loading on the Footing

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Introduction:
Jarquio and Jarquio (1983) [2] attempt to introduce the exact equations of the vertical stress under the footing by performing integration of the basic Boussinesq's (1885) [1] equation for a point load as applied to a rectangular area load with a triangular load. Unfortunately, as seen in this paper, they are incomplete and the original equation used by Jarquio and Jarquio for rectangular area is incorrect.

When referring to the article published on the web "Stresses and Surcharge Stresses due to Plane Loading on Skewed Footings" by the author http://www.facsystems.com/Surcharge.pdf the stresses on the corners of the footing can be easily derived. In this article the stresses under the corners of the footing are addressed.

Point Load:
The most important original solution was given by Boussinesq (1885) [1] for the distribution of stresses within a linear elastic half-space resulting from a point load applied normal to the surface, illustrated in Fig. 1. The results obtained were

$$\sigma_z = \frac{3P}{2\pi}\frac{z^3}{\left[r^2+z^2\right]^{\frac{5}{2}}} \qquad \text{(1)}$$

$$\sigma_r = \frac{P}{2\pi}\left[\frac{3zr^2}{\left[r^2+z^2\right]^{\frac{5}{2}}} - \frac{1-2\mu}{\sqrt{r^2+z^2}\left(\sqrt{r^2+z^2}+z\right)}\right] \qquad \text{(2)}$$

$$\sigma_r = \frac{P}{2\pi}(1-2\mu)\left[\frac{1}{\sqrt{r^2+z^2}\left(\sqrt{r^2+z^2}+z\right)} - \frac{z}{\left[r^2+z^2\right]^{\frac{3}{2}}}\right] \qquad \text{(3)}$$

$$\tau_{rz} = \frac{3P}{2\pi}\frac{z^2 r}{\left[r^2+z^2\right]^{\frac{5}{2}}} \qquad \text{(4)}$$

$$\tau_{\alpha z} = \tau_{r\alpha} = 0 \qquad \text{(5)}$$

Where μ is Poisson's ratio and other quantities in the equations are defined in Fig. 1. These stresses are the stresses which would occur in a weightless linear elastic medium. Preexisting stress due to the weight of the material must be superimposed upon these.

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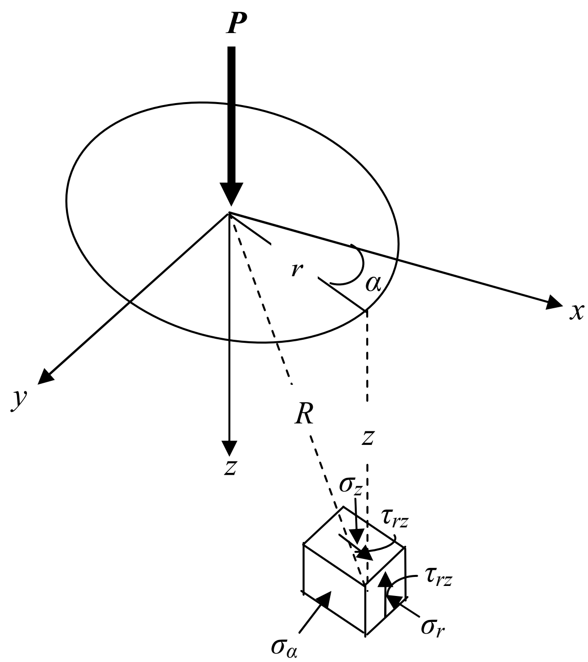

Fig. 1 Stresses in elastic half-space due to point load at the surface.

Rewriting the equation in terms of *x* and *y* using:

$\sigma_x = \sigma_r \cos^2 \alpha + \sigma_\alpha \sin^2 \alpha$, $\sigma_y = \sigma_r \sin^2 \alpha + \sigma_\alpha \cos^2 \alpha$, $\tau_{xy} = (\sigma_r - \sigma_\alpha) \sin\alpha \cos\alpha$

$\tau_{xz} = \tau_{rz} \cos\alpha$ and $\tau_{yz} = \tau_{rz} \sin\alpha$, gives two sets of equations:

Set #1

$$\sigma_x = \frac{3P}{2\pi} \frac{x^2 z}{\left[x^2 + y^2 + z^2\right]^{\frac{5}{2}}} \quad \ldots\ldots (6)$$

$$\sigma_y = \frac{3P}{2\pi} \frac{y^2 z}{\left[x^2 + y^2 + z^2\right]^{\frac{5}{2}}} \quad \ldots\ldots (7)$$

$$\tau_{xy} = \frac{3P}{2\pi} \frac{xyz}{\left[x^2 + y^2 + z^2\right]^{\frac{5}{2}}} \quad \ldots\ldots (8)$$

$$\sigma_z = \frac{3P}{2\pi}\frac{z^3}{\left[x^2+y^2+z^2\right]^{\frac{5}{2}}} \quad \ldots\ldots (9)$$

$$\tau_{xz} = \frac{3P}{2\pi}\frac{xz^2}{\left[x^2+y^2+z^2\right]^{\frac{5}{2}}} \quad \ldots\ldots (10)$$

$$\tau_{yz} = \frac{3P}{2\pi}\frac{yz^2}{\left[x^2+y^2+z^2\right]^{\frac{5}{2}}} \quad \ldots\ldots (11)$$

<u>Set #2</u>

$$\sigma_x = \frac{(1-2\mu)P}{2\pi}\left[-\frac{x^2}{\left(x^2+y^2\right)^2}+\frac{zx^2}{\left(x^2+y^2\right)^2\sqrt{x^2+y^2+z^2}}+\frac{y^2}{\left(x^2+y^2\right)^2}-\frac{zy^2}{\left(x^2+y^2\right)^2\sqrt{x^2+y^2+z^2}}-\frac{zy^2}{\left(x^2+y^2\right)\left[x^2+y^2+z^2\right]^{\frac{3}{2}}}\right] \quad \ldots\ldots (12)$$

$$\sigma_y = \frac{(1-2\mu)P}{2\pi}\left[-\frac{y^2}{\left(x^2+y^2\right)^2}+\frac{zy^2}{\left(x^2+y^2\right)^2\sqrt{x^2+y^2+z^2}}+\frac{x^2}{\left(x^2+y^2\right)^2}-\frac{zx^2}{\left(x^2+y^2\right)^2\sqrt{x^2+y^2+z^2}}-\frac{zx^2}{\left(x^2+y^2\right)\left[x^2+y^2+z^2\right]^{\frac{3}{2}}}\right] \quad \ldots\ldots (13)$$

$$\tau_{xy} = \frac{(1-2\mu)P}{2\pi}\left[-\frac{2xy}{\left(x^2+y^2\right)^2}+\frac{2zxy}{\left(x^2+y^2\right)^2\sqrt{x^2+y^2+z^2}}+\frac{zxy}{\left(x^2+y^2\right)\left[x^2+y^2+z^2\right]^{\frac{3}{2}}}\right] \quad \ldots\ldots (14)$$

The Boussinesq stresses becomes **<u>Set#1 +Set#2</u>**. Part of the reason of setting the solution in this manner, is when $\mu \approx .5$ the second set disappears. An example $\mu = .5$ for rubber.

Deriving the Stresses for Plane Loading on Footings:
To integrate Boussinesq equations using superimposition we have basically three types of loadings as in Fig. 2, 3, 4 and 5. The basic idea is to translate the point load equations in the x axis by $\bar{\xi}$ and in the y axis by $\bar{\lambda}$,as seen in the figures, then integrate with respect to $\bar{\xi}$ and $\bar{\lambda}$ over the footing $2a$ by $2b$ from $c–a$ to $c+a$ and $d–b$ to $d+b$ where the center of the footing is located at the coordinate $(c, d, 0)$.

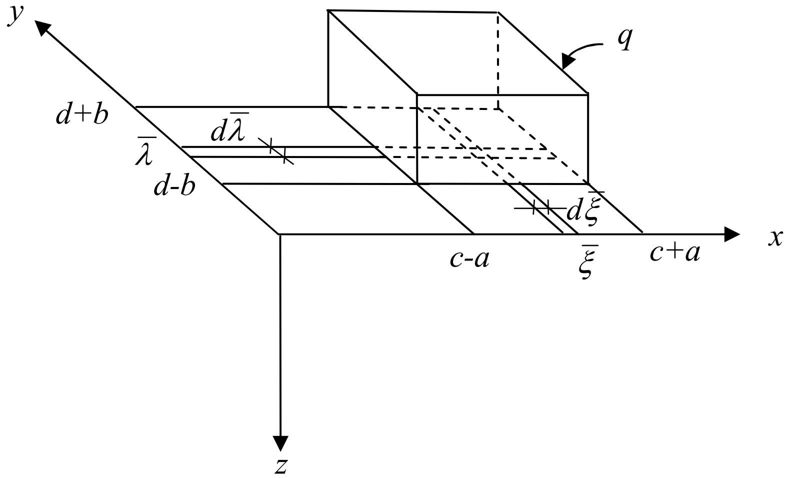

Fig. 2 Uniform Load q on 2b x 2a Footing

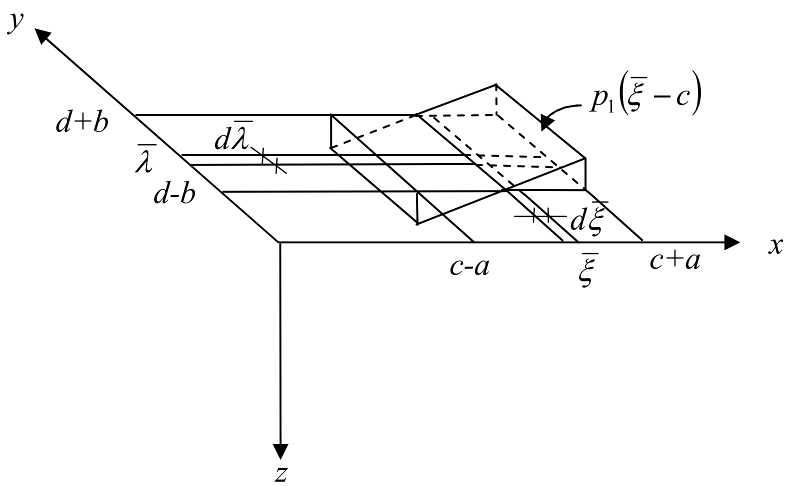

Fig. 3 Plane Load for p_1 in x on 2b x 2a Footing

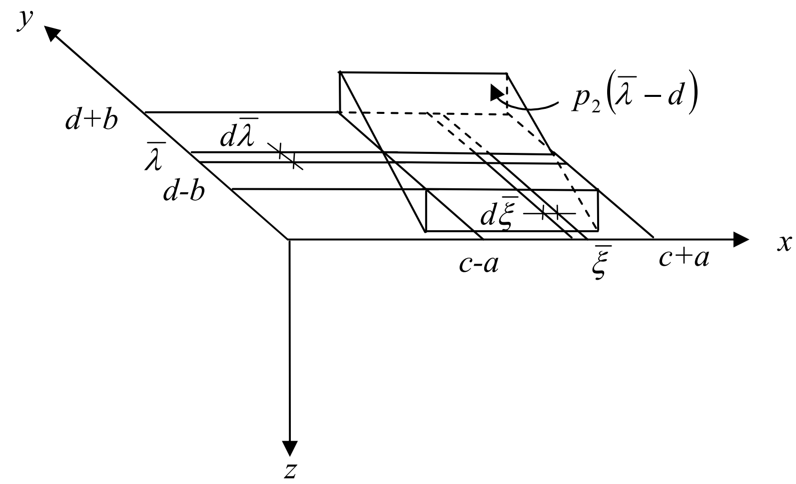

Fig. 4 Plane Load for p_2 in y on 2b x 2a Footing

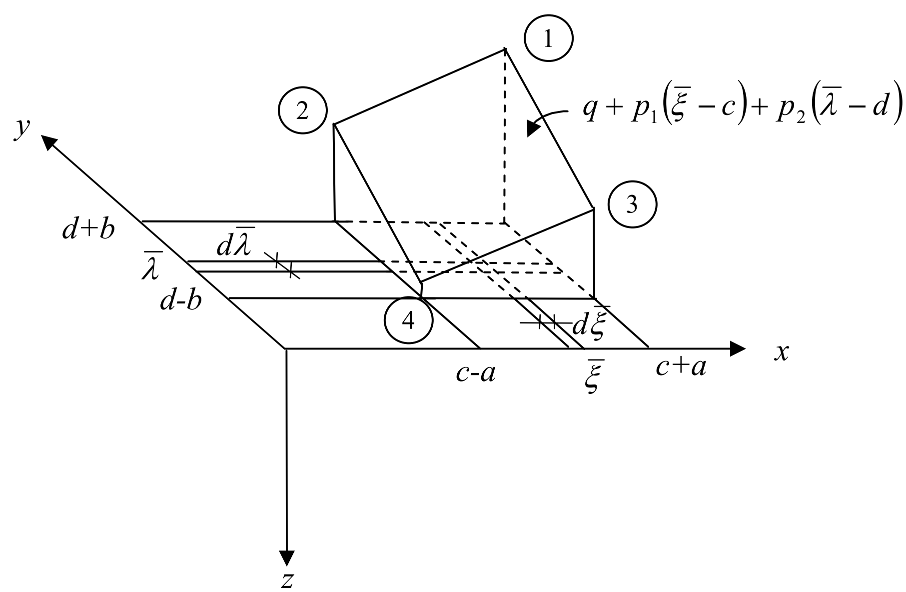

Fig. 5 Plane Load on 2b x 2a Footing

Thus we have three functions to integrate:

$z_1 = q \qquad c-a \le \bar{\xi} \le c+a$ and $d-b \le \bar{\lambda} \le d+b$

$z_2 = p_1(\bar{\xi}-c) \qquad c-a \le \bar{\xi} \le c+a$ and $d-b \le \bar{\lambda} \le d+b$

$z_3 = p_1(\bar{\lambda}-c) \qquad c-a \le \bar{\xi} \le c+a$ and $d-b \le \bar{\lambda} \le d+b$ ………………………… (15)

And the three integrations over Boussinesq equations with respect to $\overline{\xi}$ and $\overline{\lambda}$ become:

$$\int_{d-b}^{d+b} \int_{c-a}^{c+a} \sigma_i\left(x-\overline{\xi}, y-\overline{\lambda}, z\right)\, q\, d\overline{\xi}\, d\overline{\lambda} \quad \text{(16)}$$

$$\int_{d-b}^{d+b} \int_{c-a}^{c+a} \sigma_i\left(x-\overline{\xi}, y-\overline{\lambda}, z\right)\, p_1\left(\overline{\xi}-c\right) d\overline{\xi}\, d\overline{\lambda} \quad \text{(17)}$$

$$\int_{d-b}^{d+b} \int_{c-a}^{c+a} \sigma_i\left(x-\overline{\xi}, y-\overline{\lambda}, z\right)\, p_2\left(\overline{\lambda}-d\right) d\overline{\xi}\, d\overline{\lambda} \quad \text{(18)}$$

Where σ_i is the Boussinesq stress equations 6 to 14 in *x*, *y* and *z* respectively and *i* represents the *x*, *y*, *z* stress components and directions. Also the *x* is replaced by $x-\overline{\xi}$ and the *y* is replaced by $y-\overline{\lambda}$ due to the translation of the axis. Now, translate the *x* axis back by -*c* and the *y* axis back by -*d* and the center of the footing becomes the origin of the axis. The equations become

$$\int_{d-b}^{d+b} \int_{c-a}^{c+a} \sigma_i\left(x+c-\overline{\xi}, y+d-\overline{\lambda}, z\right)\, q\, d\overline{\xi}\, d\overline{\lambda} \quad \text{(19)}$$

$$\int_{d-b}^{d+b} \int_{c-a}^{c+a} \sigma_i\left(x+c-\overline{\xi}, y+d-\overline{\lambda}, z\right)\, p_1\left(\overline{\xi}-\mathrm{c}\right) d\overline{\xi}\, d\overline{\lambda} \quad \text{(20)}$$

$$\int_{d-b}^{d+b} \int_{c-a}^{c+a} \sigma_i\left(x+c-\overline{\xi}, y+d-\overline{\lambda}, z\right)\, p_2\left(\overline{\lambda}-d\right) d\overline{\xi}\, d\overline{\lambda} \quad \text{(21)}$$

Now we change variable by letting $\overline{\xi} = c+\xi$ and $\overline{\lambda} = d+\lambda$ so $d\overline{\xi} = d\xi$ and $d\overline{\lambda} = d\lambda$ and the equations become:

$$\int_{-b}^{b} \int_{-a}^{a} \sigma_i\left(x-\xi, y-\lambda, z\right)\, q\, d\xi\, d\lambda \quad \text{(22)}$$

$$\int_{-b}^{b} \int_{-a}^{a} \sigma_i\left(x-\xi, y-\lambda, z\right)\, p_1 \xi\, d\xi\, d\lambda \quad \text{(23)}$$

$$\int_{-b}^{b} \int_{-a}^{a} \sigma_i\left(x-\xi, y-\lambda, z\right)\, p_2 \lambda\, d\xi\, d\lambda \quad \text{(24)}$$

Finally we let $u = x-\xi$ and $v = y-\lambda$ so $du = dx$ and $dv = dy$ and the equations become much simpler to integrate and to keep track, and they are:

$$\int_{y-b}^{y+b} \int_{x-a}^{x+a} \sigma_i(u,v)\, q\, du\, dv \quad \text{(25)}$$

$$\int_{y-b}^{y+b} \int_{x-a}^{x+a} \sigma_i(u,v)\, p_1(x-u)\, du\, dv \quad \text{(26)}$$

$$\int_{y-b}^{y+b} \int_{x-a}^{x+a} \sigma_i(u,v)\, p_2(y-v)\, du\, dv \quad \text{(27)}$$

Now combine all the constants together and separate from the line equation yields:

$$\sigma_{i1} = \left[q + p_1 x + p_2 y\right]\int_{y-b}^{y+b}\int_{x-a}^{x+a} \sigma_i(u,v)\,du\,dv \quad \text{……… for Set\#1………………………… (28)}$$

$$\sigma_{i2} = \left[q + p_1 x + p_2 y\right]\int_{y-b}^{y+b}\int_{x-a}^{x+a} \sigma_i(u,v)\,du\,dv \quad \text{……… for Set\#2………………………… (29)}$$

$$\sigma_{i3} = -p_1\int_{y-b}^{y+b}\int_{x-a}^{x+a} \sigma_i(u,v)\,u\,du\,dv \quad \text{………………… for Set\#1………………………… (30)}$$

$$\sigma_{i4} = -p_2\int_{y-b}^{y+b}\int_{x-a}^{x+a} \sigma_i(u,v)\,v\,du\,dv \quad \text{………………… for Set\#1………………………… (31)}$$

$$\sigma_{i5} = -p_1\int_{y-b}^{y+b}\int_{x-a}^{x+a} \sigma_i(u,v)\,u\,du\,dv \quad \text{………………… for Set\#2………………………… (32)}$$

$$\sigma_{i6} = -p_2\int_{y-b}^{y+b}\int_{x-a}^{x+a} \sigma_i(u,v)\,v\,du\,dv \quad \text{………………… for Set\#2………………………… (32)}$$

The total stress becomes $\sigma_{it} = \sigma_{i1} + \sigma_{i2} + \sigma_{i3} + \sigma_{i4} + \sigma_{i5} + \sigma_{i6}$, where the following equations 33 to 60 has to be evaluated from $u = x + a$ to $u = x - a$ first then from $v = y + b$ to $v = y - b$ second. The result at $x = a$ and $y = b$, the corner circled 1 in Fig. 5 or the corner with the stress at equal $q + ap_1 + bp_2$, becomes $\sigma_{it} = \sigma_{ik}(2a,2b,z) - \sigma_{ik}(0,2b,z) - \sigma_{ik}(2a,0,z) + \sigma_{ik}(0,0,z)$, and k = 1 to 6.

$$\sigma_{x1} = \frac{\left[q + 2ap_1 + 2bp_2\right]}{2\pi}\left\{-\frac{4abz}{\left(4a^2 + z^2\right)\sqrt{4a^2 + 4b^2 + z^2}} + \tan^{-1}\left[\frac{4ab}{z\sqrt{4a^2 + 4b^2 + z^2}}\right]\right\} \quad \text{………….. (33)}$$

$$\sigma_{y1} = \frac{\left[q + 2ap_1 + 2bp_2\right]}{2\pi}\left\{-\frac{4abz}{\left(4b^2 + z^2\right)\sqrt{4a^2 + 4b^2 + z^2}} + \tan^{-1}\left[\frac{4ab}{z\sqrt{4a^2 + 4b^2 + z^2}}\right]\right\} \quad \text{……….. (34)}$$

$$\tau_{xy1} = \frac{\left[q + 2ap_1 + 2bp_2\right]}{2\pi}\left\{\frac{z}{\sqrt{4a^2 + 4b^2 + z^2}} - \frac{z}{\sqrt{4b^2 + z^2}} - \frac{z}{\sqrt{4a^2 + z^2}} + 1\right\} \quad \text{……………………… (35)}$$

$$\sigma_{z1} = \frac{\left[q + 2ap_1 + 2bp_2\right]}{2\pi}\left\{\frac{4abz}{\sqrt{4a^2 + 4b^2 + z^2}}\left[\frac{1}{4a^2 + z^2} + \frac{1}{4b^2 + z^2}\right] + \tan^{-1}\left[\frac{4ab}{z\sqrt{4a^2 + 4b^2 + z^2}}\right]\right\} \quad \text{……. (36)}$$

$$\tau_{xz1} = \frac{\left[q + 2ap_1 + 2bp_2\right]}{2\pi}\left\{-\frac{2z^2 b}{\left(4a^2 + z^2\right)\sqrt{4a^2 + 4b^2 + z^2}} + \frac{2b}{\sqrt{4b^2 + z^2}}\right\} \quad \text{………………………….. (37)}$$

$$\tau_{yz1} = \frac{\left[q + 2ap_1 + 2bp_2\right]}{2\pi}\left\{-\frac{2z^2 a}{\left(4b^2 + z^2\right)\sqrt{4a^2 + 4b^2 + z^2}} + \frac{2a}{\sqrt{4a^2 + z^2}}\right\} \quad \text{………………………….. (38)}$$

$$\sigma_{x2} = \left(\frac{1-2\mu}{4\pi}\right)[q + 2ap_1 + 2bp_2]\left\{\tan^{-1}\left(\frac{b}{a}\right) - \tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}\left(\frac{a}{b}\frac{z}{\sqrt{4a^2+4b^2+z^2}}\right) - \tan^{-1}\left(\frac{b}{a}\frac{z}{\sqrt{4a^2+4b^2+z^2}}\right) - \tan^{-1}\left(\frac{1}{z}\frac{4ab}{\sqrt{4a^2+4b^2+z^2}}\right)\right\} \quad (39)$$

$$\sigma_{y2} = \left(\frac{1-2\mu}{4\pi}\right)[q + 2ap_1 + 2bp_2]\left\{\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{b}{a}\frac{z}{\sqrt{4a^2+4b^2+z^2}}\right) - \tan^{-1}\left(\frac{a}{b}\frac{z}{\sqrt{4a^2+4b^2+z^2}}\right) - \tan^{-1}\left(\frac{1}{z}\frac{4ab}{\sqrt{4a^2+4b^2+z^2}}\right)\right\} \quad (40)$$

$$\tau_{xy2} = \left(\frac{1-2\mu}{4\pi}\right)[q + 2ap_1 + 2bp_2]\left\{\ln\left[\frac{\left(z+\sqrt{4a^2+4b^2+z^2}\right)^2}{\sqrt{4a^2+z^2}\sqrt{4b^2+z^2}}\right] - \ln\left[\frac{\left(z+\sqrt{4b^2+z^2}\right)^2}{z\sqrt{4b^2+z^2}}\right] - \ln\left[\frac{\left(z+\sqrt{4a^2+z^2}\right)^2}{z\sqrt{4a^2+z^2}}\right] + \ln 4\right\} \quad (41)$$

$$\sigma_{x3} = \frac{p_1}{2\pi}\left[2z\ln\left(\frac{2b+\sqrt{4a^2+4b^2+z^2}}{z}\right) - 2z\ln\left(\frac{\sqrt{4a^2+z^2}}{z}\right) - 2z\ln\left(\frac{2b+\sqrt{4b^2+z^2}}{z}\right) + \frac{8ba^2z}{\left(4a^2+z^2\right)\sqrt{4a^2+4b^2+z^2}}\right] \quad (42)$$

$$\sigma_{y3} = \frac{p_1}{2\pi}\left[z\ln\left(\frac{2b+\sqrt{4a^2+4b^2+z^2}}{z}\right) - z\ln\left(\frac{2b+\sqrt{4b^2+z^2}}{z}\right) - z\ln\left(\frac{\sqrt{4a^2+z^2}}{z}\right) - \frac{2bz}{\sqrt{4a^2+4b^2+z^2}} + \frac{2bz}{\sqrt{4b^2+z^2}}\right] \quad (43)$$

$$\tau_{xy3} = \frac{p_1}{2\pi}\left[z\ln\left(\frac{2a+\sqrt{4a^2+4b^2+z^2}}{z}\right) - z\ln\left(\frac{\sqrt{4b^2+z^2}}{z}\right) - z\ln\left(\frac{2a+\sqrt{4a^2+z^2}}{z}\right) - \frac{2az}{\sqrt{4a^2+4b^2+z^2}} + \frac{2az}{\sqrt{4a^2+z^2}}\right] \quad (44)$$

$$\sigma_{z3} = \frac{p_1}{2\pi}\left[\frac{2bz^3}{\left(4a^2+z^2\right)\sqrt{4a^2+4b^2+z^2}} - \frac{2bz}{\sqrt{4b^2+z^2}}\right] \quad (45)$$

$$\tau_{xz3} = \frac{p_1}{2\pi}\left\{\frac{2bz^2}{\left(2a+\sqrt{4a^2+4b^2+z^2}\right)\sqrt{4a^2+4b^2+z^2}} + \frac{4abz^2}{\sqrt{4a^2+4b^2+z^2}}\left[\frac{1}{4a^2+z^2} + \frac{1}{4b^2+z^2}\right] - z\tan^{-1}\left[\frac{4ab}{z\sqrt{4a^2+4b^2+z^2}}\right] - \frac{2bz^2}{4b^2+z^2}\right\} \quad (46)$$

$$\tau_{yz3} = \frac{p_1}{2\pi}\left[-\frac{z^2}{\sqrt{4a^2+4b^2+z^2}} + \frac{z^2}{\sqrt{4b^2+z^2}} + \frac{z^2}{\sqrt{4a^2+z^2}} - z\right] \quad (47)$$

$$\sigma_{x4} = \frac{p_2}{2\pi}\left[z\ln\left(\frac{2a+\sqrt{4a^2+4b^2+z^2}}{z}\right) - z\ln\left(\frac{\sqrt{4b^2+z^2}}{z}\right) - z\ln\left(\frac{2a+\sqrt{4a^2+z^2}}{z}\right) - \frac{2az}{\sqrt{4a^2+4b^2+z^2}} + \frac{2az}{\sqrt{4a^2+z^2}}\right] \quad (48)$$

$$\sigma_{y4} = \frac{p_2}{2\pi}\left[2z\ln\left(\frac{2a+\sqrt{4a^2+4b^2+z^2}}{z}\right) - 2z\ln\left(\frac{\sqrt{4b^2+z^2}}{z}\right) - 2z\ln\left(\frac{2a+\sqrt{4a^2+z^2}}{z}\right) + \frac{8ab^2z}{\left(4b^2+z^2\right)\sqrt{4a^2+4b^2+z^2}}\right] \quad (49)$$

$$\tau_{xy4} = \frac{p_2}{2\pi}\left[z\ln\left(\frac{2b+\sqrt{4a^2+4b^2+z^2}}{z}\right) - z\ln\left(\frac{2b+\sqrt{4b^2+z^2}}{z}\right) - z\ln\left(\frac{\sqrt{4a^2+z^2}}{z}\right) - \frac{2bz}{\sqrt{4a^2+4b^2+z^2}} + \frac{2bz}{\sqrt{4b^2+z^2}} \right] \quad (50)$$

$$\sigma_{z4} = \frac{p_2}{2\pi}\left[\frac{2az^3}{\left(4b^2+z^2\right)\sqrt{4a^2+4b^2+z^2}} - \frac{2az}{\sqrt{4a^2+z^2}} \right] \quad (51)$$

$$\tau_{xz4} = \frac{p_2}{2\pi}\left[-\frac{z^2}{\sqrt{4a^2+4b^2+z^2}} + \frac{z^2}{\sqrt{4b^2+z^2}} + \frac{z^2}{\sqrt{4a^2+z^2}} - z \right] \quad (52)$$

$$\tau_{yz4} = \frac{p_2}{2\pi}\left\{ \frac{2az^2}{\left(2b+\sqrt{4a^2+4b^2+z^2}\right)\sqrt{4a^2+4b^2+z^2}} + \frac{4abz^2}{\sqrt{4a^2+4b^2+z^2}}\left[\frac{1}{4a^2+z^2}+\frac{1}{4b^2+z^2}\right] - z\tan^{-1}\left[\frac{4ab}{z\sqrt{4a^2+4b^2+z^2}}\right] - \frac{2az^2}{4a^2+z^2} \right\} \quad (53)$$

$$\sigma_{x5} = \left(\frac{1-2\mu}{2\pi}p_1\right)\left\{ 2b\ln\left[\frac{z+\sqrt{4a^2+4b^2+z^2}}{z+\sqrt{4b^2+z^2}}\right] \right\} \quad (54)$$

$$\sigma_{y5} = \left(\frac{1-2\mu}{2\pi}p_1\right)\left\{ -2b\ln\left[\frac{z+\sqrt{4a^2+4b^2+z^2}}{z+\sqrt{4b^2+z^2}}\right] - z\ln\left[\frac{2b+\sqrt{4a^2+4b^2+z^2}}{\sqrt{4a^2+z^2}}\right] + z\ln\left[\frac{2b+\sqrt{4b^2+z^2}}{z}\right] \right\} \quad (55)$$

$$\tau_{xy5} = \left(\frac{1-2\mu}{2\pi}p_1\right)\left\{ 2b\tan^{-1}\left(\frac{a}{b}\right) + z\ln\left[\frac{2a+\sqrt{4a^2+4b^2+z^2}}{z}\right] - 2b\tan^{-1}\left(\frac{a}{b}\frac{z}{\sqrt{4a^2+4b^2+z^2}}\right) - z\ln\left[\frac{\sqrt{4b^2+z^2}}{z}\right] - z\ln\left[\frac{2a+\sqrt{4a^2+z^2}}{z}\right] \right\} \quad (56)$$

$$\sigma_{x6} = \left(\frac{1-2\mu}{2\pi}p_2\right)\left\{ -2a\ln\left[\frac{z+\sqrt{4a^2+4b^2+z^2}}{z+\sqrt{4a^2+z^2}}\right] - z\ln\left[\frac{2a+\sqrt{4a^2+4b^2+z^2}}{\sqrt{4b^2+z^2}}\right] + z\ln\left[\frac{2a+\sqrt{4a^2+z^2}}{z}\right] \right\} \quad (57)$$

$$\sigma_{y6} = \left(\frac{1-2\mu}{2\pi}p_2\right)\left\{ 2a\ln\left[\frac{z+\sqrt{4a^2+4b^2+z^2}}{z+\sqrt{4a^2+z^2}}\right] \right\} \quad (58)$$

$$\tau_{xy6} = \left(\frac{1-2\mu}{2\pi}p_2\right)\left\{ +2a\tan^{-1}\left(\frac{b}{a}\right) + z\ln\left[\frac{2b+\sqrt{4a^2+4b^2+z^2}}{z}\right] - 2a\tan^{-1}\left(\frac{b}{a}\frac{z}{\sqrt{4a^2+4b^2+z^2}}\right) - z\ln\left[\frac{2b+\sqrt{4b^2+z^2}}{z}\right] - z\ln\left[\frac{\sqrt{4a^2+z^2}}{z}\right] \right\} \quad (59)$$

Note: equations 33 to 38 matches Holl (1940) [3] for stress beneath corner of the rectangle with the exception of equation 35 a sign difference to be an error in Holl's derivation in Poulos & Davis as a typo publication. Also numerical calculations were performed for 33 to 41 and the solution matches Giroud (1970) [3]. Appendix A gives the solution in a simpler form, following Holl presentation.

To find the stress under the other corner of the footing it is easier to change the sign in p_1 and p_2 so:

for stresses on corner circled 2 in Fig. 5 change p_1 to $-p_1$ in all the equations 33 to 59
for stresses on corner circled 3 in Fig. 5 change p_2 to $-p_2$ in all the equations 33 to 59
for stresses on corner circled 4 in Fig. 5 change p_1 to $-p_1$ and change p_2 to $-p_2$ in all the equations 33 to 59

The result is summarized in the spread sheet ***stresscorner.xls*** for the stresses.

Conclusion

The solution for stresses underneath the corner of a rectangular footing due to a footing that has a plane loading equation are found by integrating Boussinesq equations.

APPENDIX A

To gives the solution in a simpler form, following Holl (1940) [3] presentation:

Let $n = 2a$, $m = 2b$, $R_1 = \sqrt{n^2 + z^2}$, $R_2 = \sqrt{m^2 + z^2}$ and $R_3 = \sqrt{n^2 + m^2 + z^2}$ and substitute in equation 33 to 59 yields:

$$\sigma_{x1} = \frac{[q + np_1 + mp_2]}{2\pi}\left\{-\frac{nmz}{R_1^2 R_3} + \tan^{-1}\left[\frac{nm}{zR_3}\right]\right\} \quad \text{(60)}$$

$$\sigma_{y1} = \frac{[q + np_1 + mp_2]}{2\pi}\left\{-\frac{nmz}{R_2^2 R_3} + \tan^{-1}\left[\frac{nm}{zR_3}\right]\right\} \quad \text{(61)}$$

$$\tau_{xy1} = \frac{[q + np_1 + mp_2]}{2\pi}\left\{\frac{z}{R_3} - \frac{z}{R_1} - \frac{z}{R_2} + 1\right\} \quad \text{(62)}$$

$$\sigma_{z1} = \frac{[q + np_1 + mp_2]}{2\pi}\left\{\frac{nmz}{R_3}\left[\frac{1}{R_1^2} + \frac{1}{R_2^2}\right] + \tan^{-1}\left[\frac{nm}{zR_3}\right]\right\} \quad \text{(63)}$$

$$\tau_{xz1} = \frac{[q + np_1 + mp_2]}{2\pi}\left\{-\frac{z^2 m}{R_1^2 R_3} + \frac{m}{R_2}\right\} \quad \text{(64)}$$

$$\tau_{yz1} = \frac{\left[q + np_1 + mp_2\right]}{2\pi}\left\{-\frac{z^2 n}{R_2^2 R_3} + \frac{n}{R_1}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sigma_{z4} = \frac{p_2}{2\pi}\left[\frac{nz^3}{R_2^2 R_3} - \frac{nz}{R_1}\right] \quad \text{(78)}$$

$$\tau_{xz4} = \frac{p_2}{2\pi}\left[-\frac{z^2}{R_3} + \frac{z^2}{R_2} + \frac{z^2}{R_1} - z\right] \quad \text{(79)}$$

$$\tau_{yz4} = \frac{p_2}{2\pi}\left\{\frac{nz^2}{(m+R_3)R_3} + \frac{nmz^2}{R_3}\left[\frac{1}{R_1^2} + \frac{1}{R_2^2}\right] - z\tan^{-1}\left[\frac{nm}{zR_3}\right] - \frac{nz^2}{R_1^2}\right\} \quad \text{(80)}$$

$$\sigma_{x5} = \left(\frac{1-2\mu}{2\pi}p_1\right)\left\{m\ln\left[\frac{z+R_3}{z+R_2}\right]\right\} \quad \text{(81)}$$

$$\sigma_{y5} = \left(\frac{1-2\mu}{2\pi}p_1\right)\left\{-m\ln\left[\frac{z+R_3}{z+R_2}\right] - z\ln\left[\frac{m+R_3}{R_1}\right] + z\ln\left[\frac{m+R_2}{z}\right]\right\} \quad \text{(82)}$$

$$\tau_{xy5} = \left(\frac{1-2\mu}{2\pi}p_1\right)\left\{m\tan^{-1}\left(\frac{n}{m}\right) + z\ln\left[\frac{n+R_3}{z}\right] - m\tan^{-1}\left(\frac{n}{m}\frac{z}{R_3}\right) - z\ln\left[\frac{R_2}{z}\right] - z\ln\left[\frac{n+R_1}{z}\right]\right\} \quad \text{(83)}$$

$$\sigma_{x6} = \left(\frac{1-2\mu}{2\pi}p_2\right)\left\{-n\ln\left[\frac{z+R_3}{z+R_1}\right] - z\ln\left[\frac{n+R_3}{R_2}\right] + z\ln\left[\frac{n+R_1}{z}\right]\right\} \quad \text{(84)}$$

$$\sigma_{y6} = \left(\frac{1-2\mu}{2\pi}p_2\right)\left\{n\ln\left[\frac{z+R_3}{z+R_1}\right]\right\} \quad \text{(85)}$$

$$\tau_{xy6} = \left(\frac{1-2\mu}{2\pi}p_2\right)\left\{n\tan^{-1}\left(\frac{m}{n}\right) + z\ln\left[\frac{m+R_3}{z}\right] - n\tan^{-1}\left(\frac{m}{n}\frac{z}{R_3}\right) - z\ln\left[\frac{R_1}{z}\right] - z\ln\left[\frac{m+R_2}{z}\right]\right\} \quad \text{(86)}$$

REFRENCES

1. Boussinesq, J., *Application des Potentiels à L'Édute de L'Équilibre et due Mouvement des Solides Élastiques,* Gauthier-Villars, Paris. 1885

2. Jarquio, R and Jarquio, V "Vertical Stress Formulas for Triangular Loading" Journal of Geotechnical Engineering, ASCE, Vol. 110, No. 1, January, 1984.

3. Poulos, H. G. and Davis, E. H. *Elastic Solutions of Soil and Rock Mechanics,* John Wiley and Sons, Inc. New York, 1974.